WORKSHEET

Riemann Sum

Class Level: 12

Kazanımlar:

İD.12.2.1.1. Bir fonksiyonun grafiği ile x-ekseni arasında kalan sınırlı bölgenin alanını Riemann toplamı yardımıyla tahmin eder.
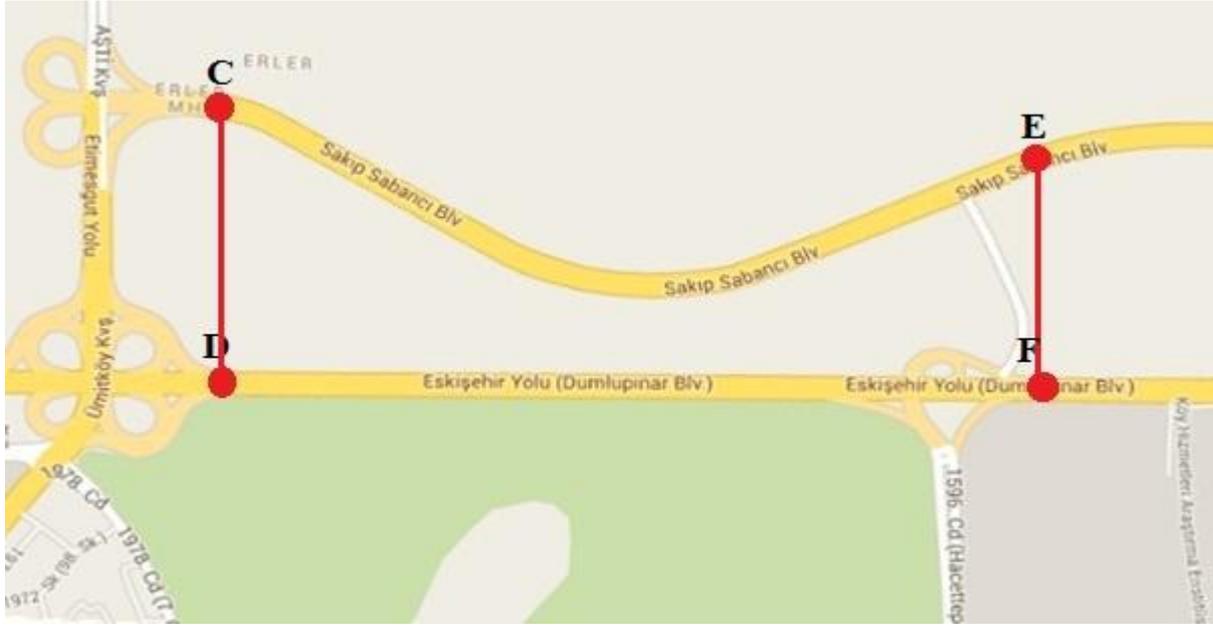
- Gerçek/gerçekçi hayat durumlarından hareketle bir fonksiyonun grafiği ile *x*-ekseni arasında kalan alanın hesaplanmasına ihtiyaç hissettirilir.

- Bir fonksiyonun belirli integrali açıklanır.

- Bilgi ve iletişim teknolojilerinden yararlanılır.

Materials: TI-84 Plus Calculator, Computer with Microsoft Office Program, Pencil, Internet, TI-84 connect cable.

1) Suppose that you have just inherited a land between Eskişehir Yolu and Sakıp Sabancı Bulvarı and this land is between CD and EF lines. How would you determine the area of your land? Write your own ideas. Make a preliminary estimation of the area of your land.

2) Look at the picture below.
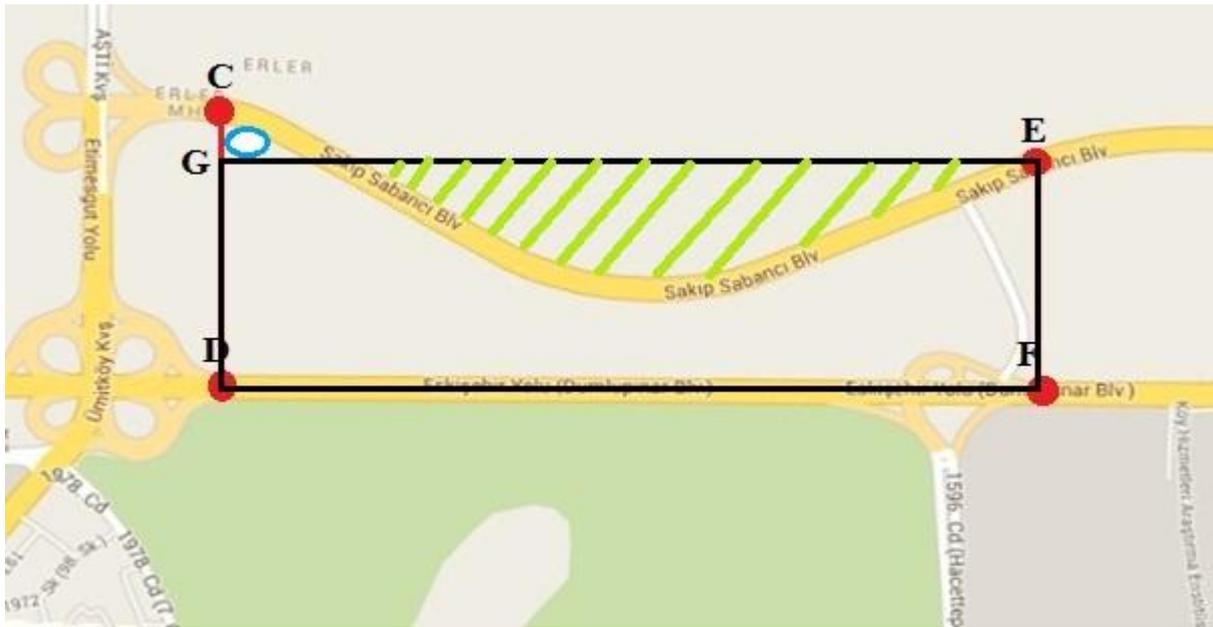
If you estimate the area of your land as the area of the DFEG rectangle it may not be the best estimate area. You may be missing the blue circled area and you may be including the green shaded area. How would you do a better estimation?

3) Look at the picture below.

If you calculate the area of your land as the sum of two rectangles with the same width above, you could do a better estimation but you still miss some area and include some area. What about if you divide your property into thirds? Does it give you a better estimate area?

➢ Approximating the area under a curve by using rectangles is known as **Riemann Sums.**

In order to find the real approximate area of your land, go to Google Maps and find the distance between two points by clicking on "Get Directions". As in the picture below, the distance between A and B is 650 meters. So, we can say that the distance between C and D is approximately 650 meters as well.
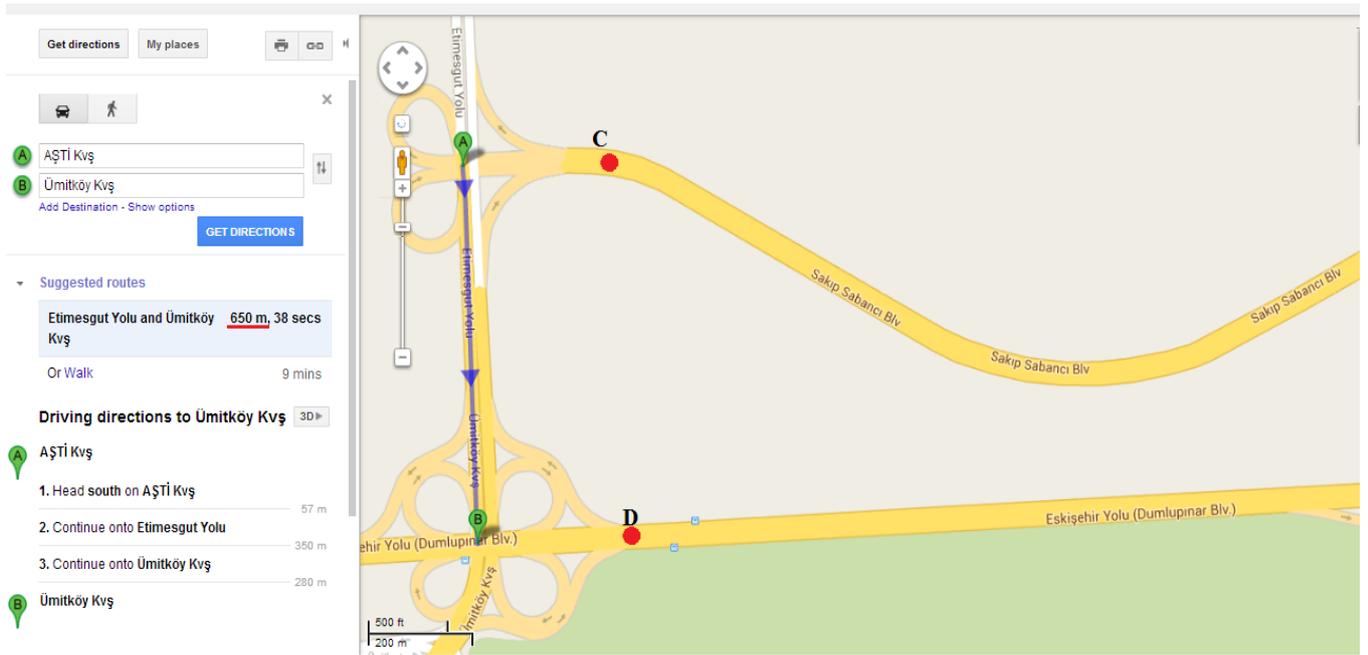
Now let's do these estimations mathematically. Call the Sakıp Sabancı Bulvarı as

$f(x),$ Eskişehir yolu as x axis and CD line as y axis. You can find the corresponding function

with Sakıp Sabancı Bulvarı by using GeoGebra software.

$$f(x) = 0.04\,x^2 - 0.82\,x + 6.81$$

As you see from the picture above, the approximate function of Sakıp Sabancı Bulvarı is

$$f(x) = 0.04 - 0.82x + 6.81$$

➢ Please pay attention to the scale! In this picture 650 m corresponds with approximately 6.5 cm.

Use your TI-84 Calculator

Now you know the function and your x interval which is (0,20). The TI-84 calculator can easily show you the Riemann sums either with left rectangles or right rectangles or trapezoids. Firstly you should download RIEMANN program from the link below and send the file to your calculator.

http://www.calcblog.com/?download=riemann

In case the web site is not available, see Appendix to find the entire code of Riemann Program.

To begin, enter the function in the **Y1** variable by pressing **[Y=]**

Press **[PRGM]**, select RIEMANN and press **[ENTER]** to run

```
█████ EDIT NEW          prgmREIMANN
1█REIMANN
```

➢ If you have your function in **Y1** enter **#1** to continue, then enter **#1** to draw a graph or **#2** to just solve numerically.

```
ENTER F(X) IN Y1

CONTINUE?
YES(1) NO(2) 1

DRAW PICTURES?
YES(1) NO(2) 1█
```

➢ For this problem enter your left endpoint as 0 and press **[ENTER]**, then enter your right endpoint as 20 and press **[ENTER]**

```
LEFT ENDPOINT? 0

RIGHT ENDPOINT?
20█
```

➢ If you want to calculate the left rectangle press **#1** and press **[ENTER],** or enter the number beside your choice. Then enter the number of slices. For this problem choose right rectangle and then choose 2 slices as the first step.
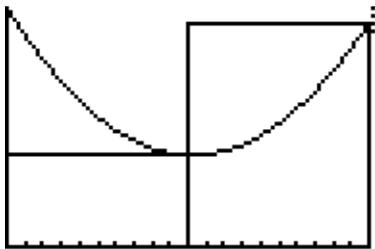
```
ENTER CHOICE:
LEFT RECT   (1)
RIGHT RECT  (2)
TRAPEZOIDS  (3)
MIDPT RECT  (4) 2

N? 2■
```

➢ Your graph will be like the picture below.

➢ By pressing **[ENTER]** you can see the sum of your rectangles.

```
RIGHT RECT  (2)   ⋮
TRAPEZOIDS  (3)
MIDPT RECT  (4) 2

N? 2
SUM=
              90.2
```

➢ By pressing **[ENTER]** you can change your choice or slice number.
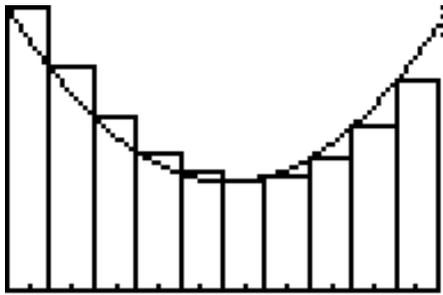
```
ENTER CHOICE    ⋮
1:CHANGE N
2:CHANGE METHOD
3:QUIT
```

➢ Now you can change the slice number by pressing **[ENTER]** when **1:** is highlighted

➢ Enter 10 for N and press **[ENTER]**

➤ Your graph will be like in the picture below.

➤ By pressing **[ENTER]** you will see the sum of the rectangles.

N? 10
SUM=
79.8

➤ **5)** Now, change your slice number as 200 and see if your graph looks like the picture

below and if the sum of the rectangles is 78.888.

6) Compare your findings and discuss which one gives you the best estimation.

Attention!!! Taking the scale into account, you have to multiply your result by 10000.

So according to 200 slices, the area of your land is approximately

$$78.888 \times 10000 = 788880 \; m^2$$

7) Find a better estimation of the area of your land by changing the slice number.

8) Find the definite integral of your function between 0 and 20 with your graphing calculator:

➢ Press **[MATH]** and choose **fnInt(** by the ▼ arrow or just press **#9** and press **[ENTER]**

$$\int_{\blacksquare}^{\cdots}(\cdots)\,d\cdots$$

➢ For the left endpoint enter **#0** and for the right endpoint enter **#20** by moving with arrows.

➢ For the inside parenthesis, press **[VARS]** and move **Y-VARS** with the right arrow.

➢ Press Enter if **1:** is highlighted.

➢ Choose **Y1** if your function is located in **Y1** variable.

```
VARS Y-VARS        VARS Y-VARS        FUNCTION
1:Window…          1:Function…        1:Y1
2:Zoom…            2:Parametric…      2:Y2
3:GDB…             3:Polar…           3:Y3
4:Picture…         4:On/Off…          4:Y4
5:Statistics…                         5:Y5
6:Table…                              6:Y6
7:String…                             7↓Y7
```

➢ Then press **[X,T,ø,n]** and choose "x" for dx and calculate the integral by pressing

 [ENTER]

9) Discuss how much your findings with Riemann sums are approximate to the definite

integral.

Finding the Riemann Sums by using Excel

We can numerically find the Riemann sums by Excel as well. We begin by opening a Workbook document in Excel. For our example, since the length of our interval is 20 and if we want 20 subintervals, we get the width of each subintervals, $x = 1$ cm. In fact, it's not hard to see that we will want to have the *x* values

$$x = 1, \quad x = 2, \quad x = 3, \quad . \quad . \quad ., \quad x = 20.$$

To get these values into the worksheet, we type 1 into cell A1 and 2 into cell A2. We then use the mouse to highlight cells A1 and A2.

Next we pick up the small black square in the lower right corner of the highlight box and drag it down column A to cell A21. Column A fills with the numbers 1, 2, 3, and so on.

To evaluate our function at all these points along the *x*-axis, we enter the function in cell B1. To do this, we highlight cell B1 and then type

$$= 0,04 * A1 \text{^} 2 - 0,82 * A1 + 6,81$$

and press **[ENTER]**. Cell B1 should now show the value of our function at *x*=1. (That value is 6.03)

Next we highlight cell B1 and then pick up the small black square in its lower right corner, and drag it down the column to cell B21. Column B fills with the values of our function at the points named in column A.

The left Riemann sum is equal to 1 times the sum of the entries in cells B1 through B20. To compute this, we choose an empty cell, highlight it, and then type

<p align="center">**=1*TOPLA(B1:B20)** or for English, **=1*sum(B1:B20)**</p>

and press **[ENTER]**. We can write an identifier for this number, such as LeftSum: to an empty cell.

The right Riemann sum is equal to 1 times the sum of the entries in cells B2 through B21. To compute this, we choose another empty cell, highlight it, and then type

<p align="center">**=1*sum(B2:B21)**</p>

And press **[ENTER].** Here's what the worksheet looks like when we have finished:

	A	B	C	D	E	F
1	1	6,03		LeftSum:	78,8	
2	2	5,33		RightSum:	80	
3	3	4,71				
4	4	4,17				
5	5	3,71				
6	6	3,33				
7	7	3,03				
8	8	2,81				
9	9	2,67				
10	10	2,61				
11	11	2,63				
12	12	2,73				
13	13	2,91				
14	14	3,17				
15	15	3,51				
16	16	3,93				
17	17	4,43				
18	18	5,01				
19	19	5,67				
20	20	6,41				
21	21	7,23				
22						
23						

10) Change the numbers of your subintervals and find a better estimate area of your land by using Excel.

11) Check if you get the same results with your graphing calculator.

Reflection

In this assignment, I aimed to prepare an inquiry based worksheet which may enhance students investigation and estimation skills. Since I am working on technology enriched inquiry based mathematics teaching on integral calculus concept for my thesis, I decided to choose a topic, Riemann Sum, which can contribute to my thesis. Doing this assignment, I came up with new ideas regarding real life examples for my thesis.

In this worksheet, I avoided to give exact formulas to students. Instead, I expected them to investigate, estimate and come up with their own formulas by using their graphing calculators and other technological tools. At the end of the investigation part, I expected students to relate the Riemann Sum with Integral. Thus, students could understand where integral concept takes place in their real life.

Finding a real life example was the first challenge for me while preparing this worksheet. Firstly, I decided to find an area on Google Maps. But it was hard to find an area which fits to my purpose, because I also wanted to find an appropriate area that students were familiar with. Further, when I found the appropriate area, I had to make some modifications on the area.

On the other hand, during the preparing process of this worksheet, I was fascinated with the improvement of technology, in particular, graphing calculators. I experienced how technology can provide students to understand the concepts in a meaningful way. Moreover, I thought that attending the workshop about "Mathematical Modeling" by Prof. Rita Borromeo Ferri helped me very much while creating this worksheet. I also experienced the importance of STEM education on students' conceptual understanding by creating this technology and engineering integrated mathematics worksheet.

Appendix

PRGM:RIEMANN

ClrHome

PlotsOff

Disp "ENTER F(X) IN Y1

Disp "

Disp "CONTINUE?

Input "YES(1) NO(2) ",G

If G=2:Stop

Disp "

Disp "DRAW PICTURES?

Input "YES(1) NO(2) ",H

ClrHome

Input "LEFT ENDPOINT? ",A

Input "RIGHT ENDPOINT? ",B

If H=1:Then

A->Xmin:B->Xmax

iPart(((B-A)/20->W

If W=0:Goto Z

Lbl U

seq(X,X,A,B,W->L5

Y□1(L5->L6

min(L6->Ymin

If Ymin>0:0->Ymin

max(L6->Ymax

```
If Ymax<0:0->Ymax

W->Xscl

iPart((abs((Ymax-Ymin)/10->Yscl

ClrList L6,L5

End

Lbl 0

ClrHome

Disp "ENTER CHOICE:

Disp "LEFT RECT (1)

Disp "RIGHT RECT (2)

Disp "TRAPEZOIDS (3)

Input "MIDPT RECT (4) ",R

Lbl 1

ClrDraw

Input "N? ",N

(B-A)/N->W

0->S:1->C

Lbl 2

If R=1:Goto 3

If R=2:Goto 4

If R=3:Goto 3

If R=4:Goto 5

Lbl 3

A+W(C-1->X

X->J:X+W->L
```

Goto 7

Lbl 4

A+CW->X

X-W->J:X->L

Goto 7

Lbl 5

If H≠1:Then

If N>5:Then

1->Z:W/2->H:A->X

Lbl 8

X+H->X:Y1□+S->S

A+ZW->X

IS>(Z,N:Goto 8

SW->S:Goto T

End:End

A+CW-W/2->X

X-W/2->J

X+W/2->L

Goto 7

A->G:G+W->G:G->V

Lbl 9

V->X:Y1->Y:V+W->X:4Y+2Y1□+S->S

V+2W->V

If V G-W->X:Y1->E

B->X:Y□1->F

(W/3)(S+E-F->S

Goto T

Lbl 7

Y▯1->K:K+S->S

If H=1:Goto D

Lbl I

IS>(C,N

Goto 2

If R=3:Then

A->X:Y1▯->P

B->X:Y▯1->Q

S+(Q-P)/2->S

End

W*S->S

Lbl T

Disp "SUM=",S

Pause

ClrHome

Lbl E

Menu("ENTER CHOICE","CHANGE N",1,"CHANGE METHOD",0,"QUIT",F)

Lbl F

Stop

Lbl D

If R=3:Then

Y▯1(L->M

```
Else:K->M

End

Line(J,0,J,K

Line(J,K,L,M

Line(L,M,L,0

If C=N:Pause

Goto I

Lbl Z

If B-A≤9:Then

.1->W:Else

1->W:End

Goto U
```